PROPER DISTANCE IN SCHWARZSCHILD FRAMES 10/16/2010

by Norman Albers nvalbers@gmail.com

Given a radial Schwarzschild metric: $ds^2 = S(cdt)^2 - S^{-1}dr^2$,
we interpret local proper distance as: $dR = \sqrt{S^{-1}}\,dr$. Thus,

$$dR = \sqrt{(1-2\text{m}/r)^{-1}}\ dr\ =\ \sqrt{\frac{r}{r-2\text{m}}}\ dr\ .$$

Let us unitize to ease manipulations: $dR\ =\ \sqrt{\frac{r}{r-1}}\ dr$.

Look at the integral from the event horizon (EH) outward to a "far radius": $R\ =\ \int_1^r dr\sqrt{\frac{r}{r-1}}$. With a little help from CRC tables this may be expressed: $R\ =\ \sqrt{r^2-r}\ +\ \frac{1}{2}\log(2\sqrt{r^2-r}\ +2\text{r}\ -1)$.

This looks consistent as it vanishes at the starting point, the EH.

Let us investigate the far limit. Express the first term as:

$$\lim_{r\to\infty} R \simeq r\sqrt{1-r^{-1}}\ .$$

Expand to second order: $\lim_{r\to\infty} R \simeq r\,(1-\frac{1}{2}r^{-1})$,

and we see an essentially constant subtraction of ½ unit in radius. Now examine the second term: $\lim_{r\to\infty} R \simeq \frac{1}{2}\log[2\text{r}-1+2\text{r}-1]\ =\ \frac{1}{2}[\log 2+\log(2\text{r}-1)]$.

This is simplified once more to: $\lim_{r\to\infty} R \simeq \log 2+\frac{1}{2}\log r$.

Adding both terms, the far limit to second order reads:

$$\lim_{r\to\infty} R = r\ +\frac{1}{2}\log r\ +\ (\log 2-\frac{1}{2})\ .$$

This is a curious result. The constant equals 0.19, always a slight (+) addition to the total radial measure. Thus total integrated proper distance is slightly more than if we had ignored the existence of the EH. That is the implication of the first term on the RHS, with the addition further of the constant, since our integration started at the EH, not at zero radius. If one plots values of R in the near-field, already at r = 1.5, the integral is 1.53 = R. At a value of r = 100, R = 102.3, with the excess slowly increasing as in the far-field limit.

One wonders if this could have cosmologic implication. I was interested here in the total radial integral but in fact this far-field is shared by *any gravitational object.* One would have to start the integral further out at the object surface for a normal, uncollapsed object, but the far-field behavior remains.