ELECTRON NEAR-FIELD CALCULATIONS: DKS AND ALBERS FIELDS COMPARED.

2/17/15 N. Albers email: nvalbers@gmail.com

The DKS fields are derived from a potential function written with Kerr coordinates: $A_0 = \frac{r}{r^2 + \cos^2\theta}$. If we take gradients in both variables, we see fields: $E_r = -\partial A_0 / \partial r$ and $E_\theta = -\partial A_0 / \partial \theta$. This is the covariant form of gradient, so we must pay attention to the Kerr metric tensor. Unlike common spherics, we have a $g_{11} = \frac{r^2 + \cos^2\theta}{r^2 + 1}$, and the $\sqrt{-g}$ is $(r^2 + \cos^2\theta)\sin\theta$.

To express contravariant form we multiply by g^{11} . We have dropped the off-diagonal term (Kerr coordinates) proportional to 'm', the Schwarzschild term, so this metric term is just the inverse of g_{11} . We get:

$$E^r = \frac{r^2 + 1}{r^2 + \cos^2\theta} \frac{\partial A_0}{\partial r} \quad , \qquad \text{and:} \qquad E^\theta = -\frac{1}{r^2 + \cos^2\theta} \frac{\partial A_0}{\partial \theta} \quad .$$

We may investigate divergence with this contravariant form. Also we may multiply the two forms for energy density, to integrate over space:

$$T = \int d^3 V \left(E_a E^a \right) \quad .$$

Section 1) ENERGY IN DKS FIELDS: Differentiating to get radial gradient,

$$\frac{\partial A_0}{\partial r} = \frac{1}{r^2 + \cos^2\theta} - \frac{2r^2}{(r^2 + \cos^2\theta)^2} \quad .$$

Let us abbreviate the cosine by writing 'c'. This is a covariant form, so to get energy density we square this and multiply by: $g^{11} = \frac{r^2 + 1}{r^2 + c^2}$.

The integration supplies a numerator term from the metric determinant, so we are integrating the square. This may be surprising but manipulation yields a form a bit easier to handle:

$$\tau = (r^2+1)\left[\frac{1}{(r^2+c^2)^2} - \frac{4c^2}{(r^2+c^2)^3} + \frac{4c^4}{(r^2+c^2)^4}\right]$$

Over one hemisphere the cosine ranges from zero to one. The radius ranges from α to infinity. All three terms reduce, given integrations by parts on the second two, to the first form, and what remains is:

$$T = 1/2\int\int dr\,dc\,(r^2+c^2)^{-2} \quad .$$

Integrating twice by *c,* this may be expressed:

$$T = 1/2\left[\frac{1}{2r^2(r^2+1)} + \frac{1}{2r^2}\int dc\,(r^2+c^2)^{-1}\right] = \frac{1}{4r^2}\left[\frac{1}{r^2+1} + r^{-1}\tan^{-1} r^{-1}\right] \quad .$$

It remains to do the integration over *r*.

Let us consider terms near the inner limit of integration. Radius value is small compared to 1, so we are effectively, to first order, integrating a form:

$\tau = \frac{1}{4r^2}\left[1 + \frac{\pi}{2r}\right]$. The first term seems welcome, as it integrates to inverse *r,* although the coefficient is too small by the factor of 4. However the second term yields another order of inverse *r* which is not welcome !!!

Let us examine the θ-component for energy. Look at these fields developed from the potential function:

$$-E_r^{DKS} = R^{-2} - 2r^2R^{-4} = [R^2-2r^2]R^{-4} = (-r^2+\cos^2\theta)R^{-4} \quad ; \quad -E_\theta^{DKS} = rR^{-4}(2\cos\theta\sin\theta) \quad .$$

Multiplication by g^{22} yields contravariant form in θ, and we examine the second term:

$$-E^{\theta DKS} = r\,R^{-6}(2\cos\theta\sin\theta) \quad .$$

Once again when we construct the integrand, another factor of R^2 appears:

$$\tau = R^{-8}r^2(2\cos\theta\sin\theta)^2\sin\theta \quad ,$$

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$$T = \int\int dr\, d\theta\, r^2 (r^2 + \cos^2\theta)^{-4} (2\cos\theta\sin\theta)^2 \sin\theta \quad .$$

This may look horrendous but is quite soluble. Start with the angular integration, because it yields, in the end after several integrations by parts,

$$\int \frac{dc}{r^2 + c^2} = \frac{1}{r} \arctan \frac{c}{r} |_0^1 \quad .$$

At the lower limit things vanish, but at the upper one, *c*=1, think of how *r* ranges. For large radii, things vanish, but for small, the arctan is π/2, so the result is π/(2r). I am leaping a bit but this works as we go to integrate also in *r*. I use such seat-of-the-pants analysis: to integrate by *r*, we include the factor on top:

$$\int dr\, r \arctan 1/r \quad .$$

Now at small *r,* we can see the integrand goes as π*r*/2, and the integral has a positive exponent in *r* and so is quite weak, evaluated at $r = \alpha$.

Refreshing the context here, with unitized elements, the classical result for energy would be 4π/*r,* although I do not bother here with the 2π integration around the featureless circle. I had formerly thought the radial contribution was small, but results here offer a chaotic result.

Section 2) DIVERGENCE : The divergence operator is:

$$\nabla \cdot E = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (\sqrt{-g}\, E^a) \quad .$$

The simplest calculation to present is on my proposed near-fields, constructed for zero divergence: $E^r = R^{-2} + k\alpha R^{-2} r^{-1} P(\theta)$, and $E^\theta = k\alpha R^{-2} r^{-2} \sin\theta\cos\theta$. I have defined: $P(\theta) = 3\cos^2\theta - 1$. Notation for the radial term is simplified:

$$R^2 \equiv (r^2 + \cos^2\theta) \quad .$$

Divergence is: $$\nabla \cdot E = R^{-2}\frac{\partial}{\partial r}(R^2 E^r) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}(R^2 \sin\theta\, E^\theta) \quad .$$

We can see the *R*-terms cancel in both parts and this is what makes my divergence simple. Rewrite:

$$\nabla \cdot E = R^{-2}\frac{\partial}{\partial r}[1 + k\alpha r^{-1} P(\theta)] + \frac{k\alpha}{r^2 R^2 \sin\theta}\frac{\partial}{\partial \theta}[\sin^2\theta \cos\theta] \quad .$$

Bingo. This is zero.

Section 3) ENERGY IN ALBERS FIELDS: Energy density in my near-fields will be, acknowledging the orthogonality of components:

$$T = \int \sqrt{-g}\, dr\, d\theta\, E^a E_a \quad .$$

Expressing the covariant field components:

$$E_r = (r^2+1)^{-1} + k\alpha r^{-1}(r^2+1)^{-1} P(\theta) \quad \text{and} \quad E_\theta = k\alpha r^{-2} \sin\theta \cos\theta \quad .$$

The integrand for energy of radial terms is:

$$\tau = \sqrt{-g}\, E^r E_r = R^2 \sin\theta[(r^2+1)^{-1} + k\alpha r^{-1}(r^2+1) P(\theta)][R^{-2} + k\alpha r^{-1} R^{-2} P(\theta)] \quad .$$

Arranging terms, $$\tau = [(r^2+1)^{-1} + k\alpha r^{-1}(r^2+1)^{-1} P(\theta)][1 + k\alpha r^{-1} P(\theta)] =$$
$$(k\alpha)^2 r^{-2}(r^2+1)^{-1} P^2 + k\alpha r^{-1}(r^2+1)^{-1} P + (r^2+1)^{-1} \quad .$$

The readers may entertain themselves expressing these integrals, but first let us develop some analytic sense of the results. Integrating over θ reduces a total only so much; we ask what terms are significantly large. Any (r^2+1) has its near-field intensity cut back, clearly: as 'r' gets small, it does not drive the denominator to zero. Thus we shall look at the first term. Evaluation at the inner limit of $r = \alpha$ is the essence and only this term has significant magnitude.

Remember that the added field components share an arbitary constant multiplier in my scheme, 'b'. This will be the FSC times some multiplier, 'k', and there are two orders of it in the numerator. With two such quantities, the integration yielding something in the neighborhood of 1/*r* is reduced to $k^2\alpha$. Let us look at the θ -component of energy.

The integrand of this part reads:

$$\tau = d^3 V\, E^{\theta} E_{\theta} \approx R^2 \sin\theta\, r^{-4} R^{-2} \sin^2\theta \cos^2\theta \ ,$$

$$\tau \approx r^{-4} \sin^3\theta \cos^2\theta \ .$$

Once again the integration in θ will reduce the total but not greatly, but we see a higher order of inverse *'r'* which is very powerful here. The sines and cosines integrate fairly easily, reducing totals as expected, but the crux is the integral of $\int dr\, r^{-4} = -1/3\, r^{-3}$. The (-1) sign in inconsequential since we analyze at the lower limit. The numerator, as before, has $k^2\alpha^2$ but this is trumped by the extra order of *'r'* in the denominator. Only such a term has the strength to yield adequate energy by the classical energy radius.

This is such an important calculation that I detail it:

$$T = \int\int dr\, d\theta \sin^3\theta \cos^2\theta\ r^{-4} \quad \text{or} \quad T = -\int\int dr\, dc\, (1-c^2) c^2 r^{-4} \ .$$

Limits on *r* are, r_e to infinity, and the cosine ranges from -1 to 1 over the spheric volume. Thus, $T = \frac{1}{3r^3}[2/3 - 2/5]$, which is $T = \frac{4}{45\, r^3}$.

So the constants “eat up” one Oom.

Recall that an arbitrary constant is available as coefficient, equally applied to the radial and the angular additions of electric field. Thus it will appear squared in the energy totals, at least aside from cross-terms, and also the “homogeneous” part of the radial field. These contribute not so much to energy.

Figure the constant as: $K = k\alpha$. We need a net result of: $T = \frac{2}{\alpha}$. (I leave out final integration over φ. Our circles are assumed featureless.)

If now we say let the inner radius equal α, and equate energy forms:

$$\frac{2}{\alpha} = \frac{4k^2\alpha^2}{45\alpha^3} \quad ,$$

then we see that $k^2 = 22.5$, so k = 4.7 .

This is the answer, if we add field components with equal *K.* There is a second clear mathematic possibility. Classical energy radius is conceptually simple. It answers the question, at what inner radius have we accounted for the particle's total mass/energy in the electric field??? Certainly we can ascribe experimental verity to the Compton wavelength, and it is 4π larger than Kerr's AM radius, 'a'. If we multiply 'a' by the fine structure constant we have the classical radius. So what? I think this is not such an experienceable quantity, and so let us perturb it smaller. It was conceived as that radius in the case of business as usual IN A FLAT SPACE, where energy is accounted for. The Kerr spin fields weaken electric near-fields, by altering the geometry of spacetime here. Thus is may be reasonable to ask, integrating a bit further inward, can we produce enough energy?

Since my added θ-component yields, under energy integration, three inverse orders of *r*, this is powerful algebra. Observe a relaxed limit of

integration: $$T = \frac{4}{45\,r^3} = 2\alpha^{-1} \quad .$$

Now we apply a constant in a different manner, leaving the fields with a coefficient of 1, but writing the radius itself, with a small constant:

$$\alpha^{-1} = \frac{(\alpha)^2}{22.5(K\alpha)^3} \quad .$$

We see that $K^3 = 1/22.5$, so K = 1/ 2.7 = 0.37 . This says, if we write the added field terms with appropriate orders of *r* and α, we can integrate in a bit further, and get a good energy total. Results are quite distinct from the DKS analysis, as here the strength is manifested by the θ-component.

Only after doing this do I realize I defeated one purpose here. Leaving the added field magnitudes unchanged with no coefficient other than α, the radial field strength does go to zero at the original classical radius, and then I go in further. Thus it is possible to ask, what field might avoid such pathology? Observe that the inverse-cube of the inner limit is what matters, and also the square of the field strength. How may we leave the overall fraction in brackets:

$$E^r = R^{-2}[1+\alpha\frac{P(\theta)}{r}] \;\; ,$$

just equalling zero at the inner limit, where $P(\theta) = -1$ at the ring edge? With a higher field strength, imagine such a factor on top. The analyzed fraction must balance its coefficient of 1/22.5, so in fact if we relax the inner classical radius by a factor of 22.5 , then we need only increase magnitude of the field by the inverse of that number. This is the farthest I can go with this idea.