Casimir and Atomic Dipole Forces

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Casimir himself was considering interatomic Van der Waals forces to explain the attraction between two good and very flat conducting sheets.[1] I offer a rather simple path to see how well this models the physics usually presented by a quantum vacuum analysis.

A conductor is a reflector of energy, up to the plasma frequency of the available conduction electrons. Any EM text shows the field of a charge near a plate. The field lines are bent because the conductor will redistribute charge laterally until there are no sidewards components. Thus it is just as if there was an image charge in back of the plate, one of opposite sense.

We know the electric force of one charge on another varies as inverse-squared radius. Furthermore, the field from a dipole varies as inverse-cubed distance. It is critical to realize this describes the force from a dipole *upon a test charge.* It is yet a further question to ask of the force from one dipole upon another. Following the same analyses, it shouldn't be surprising that this force varies as the inverse-fourth power of separation. Now just as a monopole charge creates its image, so will a dipole near a conductor create its image charge of opposite sense. If the surface of the metal is characterized by dipole manifestations, they should induce opposite images in a nearby plate.

If each lattice atom contributes one conduction electron which is spread fairly freely throughout between atoms, then we might expect there to be net “+” charge at the metal atom, with a spread “-” charge everywhere. Let us start by considering each lattice atom as being half of a dipole together with a conduction electron. This will be a first-order approximation of the total neutral substance. Characterize the dipole as charge 'e' times the charge separation, which we take as half the interatomic distance, and say: $p=eL/2.$

Let us add the cumulative force of dipoles from each atom. Figure the number of sites,'N', per unit volume. Here I approximate the substance as a cubic collection of average interatomic distance 'L'. Using Avogadro's number, and the mass density and the atomic weight, we may write:

$$N = (\textit{mass density})\ x\ (\textit{Avogadro}) / (\textit{atomic wt.})$$

The number of atoms showing per unit area on a face, is the product: *NL.* In fact, we may now express atomic separation as: $L^3 = 1/N$. Thus the face density goes inversely as the square of 'L'. We expressed the dipole just above. Our physics says that there is an attractive force between each dipole and its image, so let us calculate this precisely.

Consider two vertical dipoles as above. They should be directly parallel, as they would be in a mirror. Let us write the sum of forces on one particle, given an opposite image acting as if it were behind the mirror. The attraction to its opposite is a bit stronger since there is a small diagonal involved with the repelling reaction. Thus:

$$F_1 = \frac{e^2}{4\pi\epsilon_0}\left[\frac{1}{(2\mathrm{a})^2} - \frac{\cos\theta}{(2\mathrm{a})^2 + (L/2)^2}\right] \ .$$

Here $\cos\theta$ expresses the small angle to the lower repelling source, and we look at the sideways component.This is simply equal to:

$$\cos\theta = 2\mathrm{a}/\sqrt{((2\mathrm{a})^2 + (L/2)^2)} \ , \quad \text{so we get:}$$

$$F_1 = \frac{e^2}{16\pi\epsilon_0 a^2}\left(1 - \left[1 + \left(\frac{L}{4\mathrm{a}}\right)^2\right]^{-3/2}\right) \ .$$

We look at many atomic lengths separation so we may expand in 'L/a'. Then:

$$F_! = \frac{e^2}{256\pi\epsilon_0 a^2}(3/2)\frac{L^2}{a^2} \ .$$

The other particle experiences the same attraction, so we may double this.

Now, recall that there are L^{-2} dipoles per unit of face area, so the total force density loses any dependence on the detail, 'L' ! This seems strange at first, until you think of how a larger count per area makes each dipole smaller, and separation counts. We have figured the attraction per unit area on one plate from its images. In fact the other plate induces its images onto this plate, and so we may put in yet another factor of two:

$$F = \frac{3e^2}{128\pi\epsilon_0 a^4}. \quad .$$

This is the force per unit area of plate.

Now let us look at the expression from the quantum analysis,:

$$F_C = \frac{\pi^2 \hbar c}{240 a^4} = \frac{\pi hc}{480 a^4} \quad .$$

Reaching up our sleeve to pull out a substitution for Planck's constant as expressed in the Fine Structure Constant, where: $\alpha = \frac{e^2}{2\epsilon_0 c h}$,

we may now write: $$F_C = \frac{\pi}{960} \frac{e^2}{\epsilon_0 \alpha a^4} \quad .$$

Let us compare the coefficients of the very similar expressions. The dipole model has $\frac{3}{128\pi}$, while the quantum model has $\frac{\pi}{960\alpha}$. The inverse of α is the number 137, so we compare 0.0074 (dipoles) to 0.45 (quantum vacuum). Though similar in functional dependence, the dipole forces are smaller by a factor of about 60. I predict, though, around the conduction plasma frequency in the ultraviolet, even silver ceases to reflect. Thus if we can experiment at such close separations we should see the Casimir force curve level out. Then at half this separation, at fourth power dependence, the dipole force is stronger by a factor of 16, so it is becoming significant, and indeed at

1/3 of the separation has a relative magnitude of 81, while the Casimir force peaked at about 60. Thus the curve should again rise after a flat spot.

Looking at the plasma frequency for silver from the formula:[2]

$$\omega = \sqrt{\frac{Ne^2}{\epsilon_0 m_e}} \quad ,$$

where 'N' is atomic density, and the denominator has electron mass. Using specific gravity, and silver's atomic weight, I calculate a frequency:

$$\nu = \omega / 2\pi = 2.2\text{E}15 \quad .$$

If we divide this into the speed of light, the equivalent photon wavelength is about 1370 Angstroms. If we can control to such a separation, there should be apparent here a flat spot in the rising force curve.

[1] Jaffe, R. (2005). "Casimir effect and the quantum vacuum". *Physical Review D* **72** (2): 021301.

[2] Feynman Lectures in Physics, Vol. II, p.7-7.