Title: Vacuum Solutions and Divergence in Kerr geometry

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Abstract: The Kerr solution of the Einstein field equations (EFE's) [1] presents a curved spacetime geometry, in which the flat-space rules of field divergence do not hold. Tensor math is needed to express possibilities. Energy integrations show the failure of Debney, Kerr, Schild (DKS) fields [2] in producing correct energy totals, and the success of the proposed Albers fields. Expressed in Kerr's geometry, the Albers electron field was plotted [3] and reveals the conformation of field components, in Cartesian form. There are individual electric field lines as well as a color-coded scale of energy density, stated at the bottom. Also stated is the angular spacing of individual lines, since in the far-field they are identified as radial "spokes". These plots give insight into the structure and behavior of the electron fields.

1. Schwarzschild and Kerr solutions

In 1963 Roy Kerr solved the EFE's by expanding the Schwarzschild solution. The equations produce a final Laplacian form, which is solved by inverse radius. This may be offset by any constant, and still be a solution, and Kerr chose an imaginary constant to offset one axis. This brilliant move allows us to identify angular momentum as a field source .

The Schwarzschild solution gives us a metric form, in spherical coordinates, of:

$$ds^2 = Sdt^2 - 1/S\, dr^2 \text{ - …(angular terms)} \qquad (1)$$

with the usual flat-space angular terms. We set the speed of light equal to 1. Here S is defined as $S = 1 - 2m/r$, where $m = GM/c^2$ with a central source of mass/energy 'M'. If we allow the *z*-axis to be offset by some imaginary constant, 'a',

$$ds^2 = dt^2 - \{r^2 + a^2 \cos^2\theta\} / \{ r^2 + a^2\}\, dr^2 - (r^2 + a^2 \cos^2\theta)\, d\theta^2 \text{ - ,,,} \qquad (2)$$

Here we leave out terms proportional to '*m'* since in the realm of elementary particles, this is a vanishingly small quantity. For electrons the Schwarzschild radius is 10^{-57}m, and neither I nor many other folks are concerned about physics at such small dimension It is of little concern that the theory produces something not useful, here. We may also simplify by setting *a*=1, and identity metric terms as:

$$g_{00} = 1, \quad g_{11} = (r^2 + cos^2)/(r^2 + 1) \;, \quad g_{22} = (r^2 + cos^2\theta) \;. \tag{3}$$

Tensor calculus gives form for divergence of a field:

$$DIV = 1/\sqrt{-det}\;\; \partial/\partial x^{\,i}\; (\sqrt{-det}\;\; F^{\,i}) \quad , \tag{4}$$

In the Kerr geometry, there is a non-trivial determinant whose root is given by

$$\sqrt{-det} = (r^2 + cos^2\theta)\, sin\theta \,. \tag{5}$$

Note the flat-space form has no cosine.

2. DKS complexification

In 1969 the DKS paper on complexification of the field equations was published, and presented an electron field solution. First we quote the mathematical characterization of divergence in a curved space: “An integrable space [exists] if and only if the Riemann tensor is identically zero.” [4]. I add, “uniquely integrable”. We are looking at any vector field under parallel displacement. This complicates the calculus between a field and any potential form supposed to generate it; there is not an unique relationship. One may choose to start with a potential expression and take gradients to get fields. The reverse operation IS NOT UNIQUE, there is path dependence. There may be fields with zero divergence that do not trace back to an unique potential form.

Not all complexifications are useful! The electric near-field produced by the DKS methodology may be integrated for its total energy, by constructing the integral of field squared. We must correctly include the relevant form of differential volume element:

$$d^3V = \sqrt{-det}\;\; d^3x^{\,i} \;. \tag{6}$$

Total electric energy will be: $\xi = 1/2 \int E^2\, d^3V$;

the geometry gives two near-field components, one in r and another in θ. The DKS fields are:

$$E_r = (-r^2 + c^2)/(r^2 + c^2)^2 \quad \text{and} \quad E_\theta = 2r\; sin\,\theta\; cos\theta/(r^2 + c^2)^2 \,. \tag{7}$$

From this point I write the cosine as *c.* The only strength in this field is the radial component as the reader will see after going thru this account. In a tensor space we write an invariant form for the square: $E^2 = E^i E_i$, acknowledging the DKS field is a covariant construction; it is taken as the gradient of the real part of complexified inverse radius ! Again we need to follow rules of tensor calculus and raise the index on the field: $E^i = g^{ik} E_k$ which in the present case is: $E^r = E_r\ g^{11}$. Since we have dropped off-diagonal terms as vanishingly small, we may simply invert the metric coefficients. Thus,

$$\xi = 1/2 \int E_r{}^2 (r^2 + 1)\, d^3V \quad .$$

We can see a cancellation in one order of $(r^2 + c^2)$, and so:

$$\xi = 1/2 \int [(-r^2 + c^2)^2 /(r^2 + c^2)^4]\, (r^2 + 1)\, sin\theta\, dr d\theta d\varphi \ . \qquad (8)$$

Our interest lies in the near-field limit of *r* ; in the far everything reduces to just one over *r*-squared. Readers may convince themselves that the numerator term in r^2 is much smaller than that with 1, so we ignore it. The relevant part is:

$$\xi = 1/2 \int(r^4 + c^4 - 2r^2c^2)/(r^2 + c^2)^4\, dr(-dc)d\varphi \ . \qquad (9)$$

All these integrations contain $sin\theta\, d\theta$ so we may write this as $-dc$. The integration is approachable by parts, with details given in Appendix A. The result is unsatisfying:

$$\xi = 1/4r(1 + \pi/4r)\ , \qquad (10)$$

to be evaluated at the inner 'energy radius', since in the far there is, as before, no contribution. In this result, the first term is too small by a factor of 4, from the unitized 1/*r* we expected, but the second term is too large. I have personally challenged Alexander Burinskii to defend his support of these fields - this is yet to be forthcoming.

3.1 Albers fields

The DKS fields have no great intensity close in except near the ring equator edge, and the field component in θ is fairly weak at mid-θ because of the presence of *r* in the numerator. The Albers fields are constructed to process well under the divergence operator, and are thus necessarily contravariant. Rearranged radial dependencies produce a different set of fields, still having zero divergence, as do the DKS, though these integrate well:

$$E^r = [1/(r^2+c^2)]\,[1+kr^{-1}P(\theta)] \quad \text{and} \quad E^\theta = k\,sin\theta\,cos\theta/r^2(r^2+c^2)\,. \tag{11}$$

Forming the invariant expression for energy density, as before, but starting with this contravariant field, we will see strength is now in the angular component.
Orders of the polynomial (r^2+c^2) vanish, and here is no such intensity at the ring edge; our "PAC-MAN" shaped region is blue and theirs is red! Looking at the DKS radial field term, we see a curious null point where $r^2 = c^2$, the "Tinker Bell" region, or point). The point on the central spin axis, the "z"-axis, where *r*=1, both field components are zero. The first version of Albers fields showed a similar "Tinker Bell" point just outside the ring edge! We may let go of our use of the classical energy radius here, and clearly this is not a classical situation. The inner limit of our energy integration may be chosen to make more sense.

There is another arbitrary constant multiplying both added field components. If they are divergence-free fields, contributions from *r* and θ cancel, and can both be multiplied by a common factor. Thus the Albers field construction presents two arbitrary constants. This allowed the Tinker Bell point to be sequestered just at the ring edge ! Each added field component is multiplied by *k*, and also the inner integration limit is *k*. The denominator of the integrand contains r^4 from being squared, and in the numerator is $k^{2,}$. Thus the integration result goes as k^2/r^3 .

3.2 The expressions

As stated in eq. (11), the radial component of the Albers electron field is given by

$$E^r = (r^2+c^2)^{-1}\ [1\ +\ k/r\,P(\theta)]\,, \tag{12}$$

where P is the second Legendre polynomial of the cosine of polar angle θ, $P(\theta)\ =\ 3cos^2\theta$ - 1. The value of *k* is the fine structure constant reduced by a factor of 22.5. This factor comes about from the integration of energy density of the θ term. The electric field tangential component is given by:

$$E^\theta\ =\ k\,sin\theta\,cos\theta/r^2(r^2+c^2)\quad . \tag{13}$$

When squared, its numerator contains: $sin^2\theta\,cos^2\theta = cos^2\theta(1\ -\ cos^2\theta)$. The integrand is written: $c^2(1-c^2)\,dc\ =\ (c^2\ -\ c^4)dc$, and both factors of (r^2+c^2) in the denominator are cancelled out, one by lowering one index, and the other by including d^3V . Thus we simply perform and evaluate the integral of cosines. This gives:

$$\int_0^1 dc\,(c^2 - c^4) = 1/3 - 1/5 = 2/15 \ . \qquad (14)$$

There appears a factor of ⅓ from the *r*-integration, so we see 2/45, or 1/22.5.

3.3 Divergence

Define D = $\sqrt{-\,det}$ and divergence is: $DIV = D^{-1}\,\partial/\partial x^i\,(D\,E^i)$. We find ourselves with two electric field components as of 10^{-13}m. The Kerr reference frame is rotated with respect to the external Cartesian frame, and furthermore, a second component is produced, in θ. In both the DKS fields and the Albers fields each component produces a divergence, but *they are equal and opposite.* Each field has zero divergence. Plots of these fields are shown below. This is the manifestation of the non-integrability of a curved spacetime. The Albers fields cannot be traced back with an “un-gradient” operation, to a single potential form, and the DKS fields can. We should move on starting with good fields, and not concern ourselves with having no unique potential form.

3.4 The plots

After much learning combined plots were constructed with electric field lines and field density. The field lines were computed by integrating E^r and E^θ inward from regularly spaced starting points. The outer distance in Figure (1) is ~3*a*. In absolute terms,

$$a = GJ/c^3 m \quad , \qquad (15)$$

where *m* is the Schwarzschild radius, $m = GM/c^2$. Thus,

$$a = J/Mc \quad , \qquad (16)$$

and for electrons is about 10^{-13}m ! Compared with the Schwarzschild radius of 10^{-57}m, this term is almost macroscopic. The intrinsic angular momentum is $\hbar/2$.

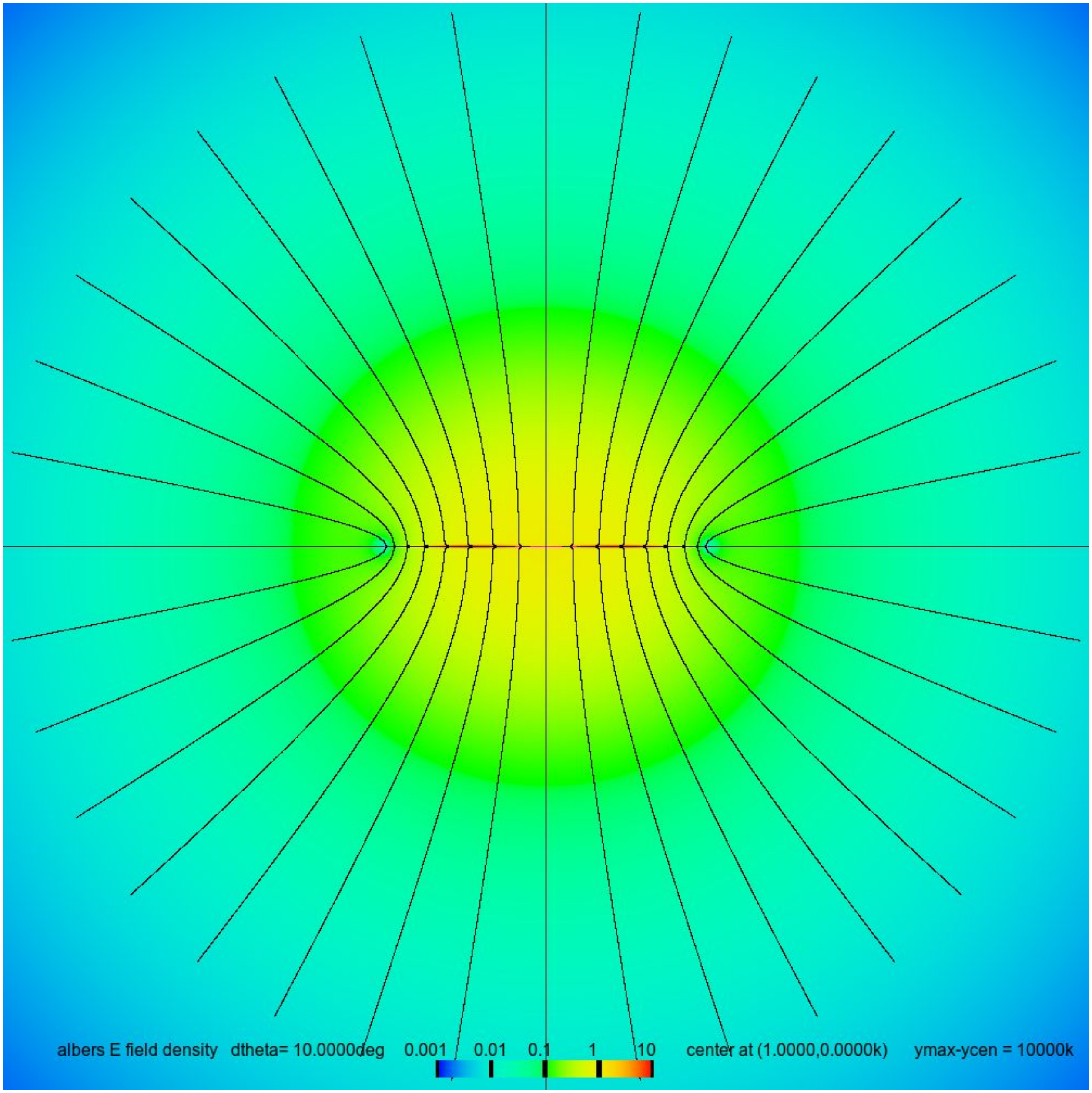

Fig 1. Electron field orientation (lines) and density (colors). The origin is in the center and x and y vary between +/- ~3 times the Kerr radius. The x-axis is in the equatorial plane and y represents the polar axis.

The integration starting points are specified in Cartesian azimuthal intervals depending on the plot magnification. For each plot frame we perform the line integration inward considering the electric field in Kerr metric coordinates using eqs. (12), (13). The result is transformed into Cartesian coordinates for display. The entire series of larger magnifications, including zooming animations may be seen at [3]. We see individual electric field lines exhibiting a very near-field behavior of sweeping at an oblique angle into the source layer at r_e. This layer is by Alexander Burinskii's theory, "very thin" [5] at the boundary of a false vacuum regime inside. Any charge layer must, by electrodynamic constraints, be very thin. If it weren't, concentric layers would magnetically coalesce.

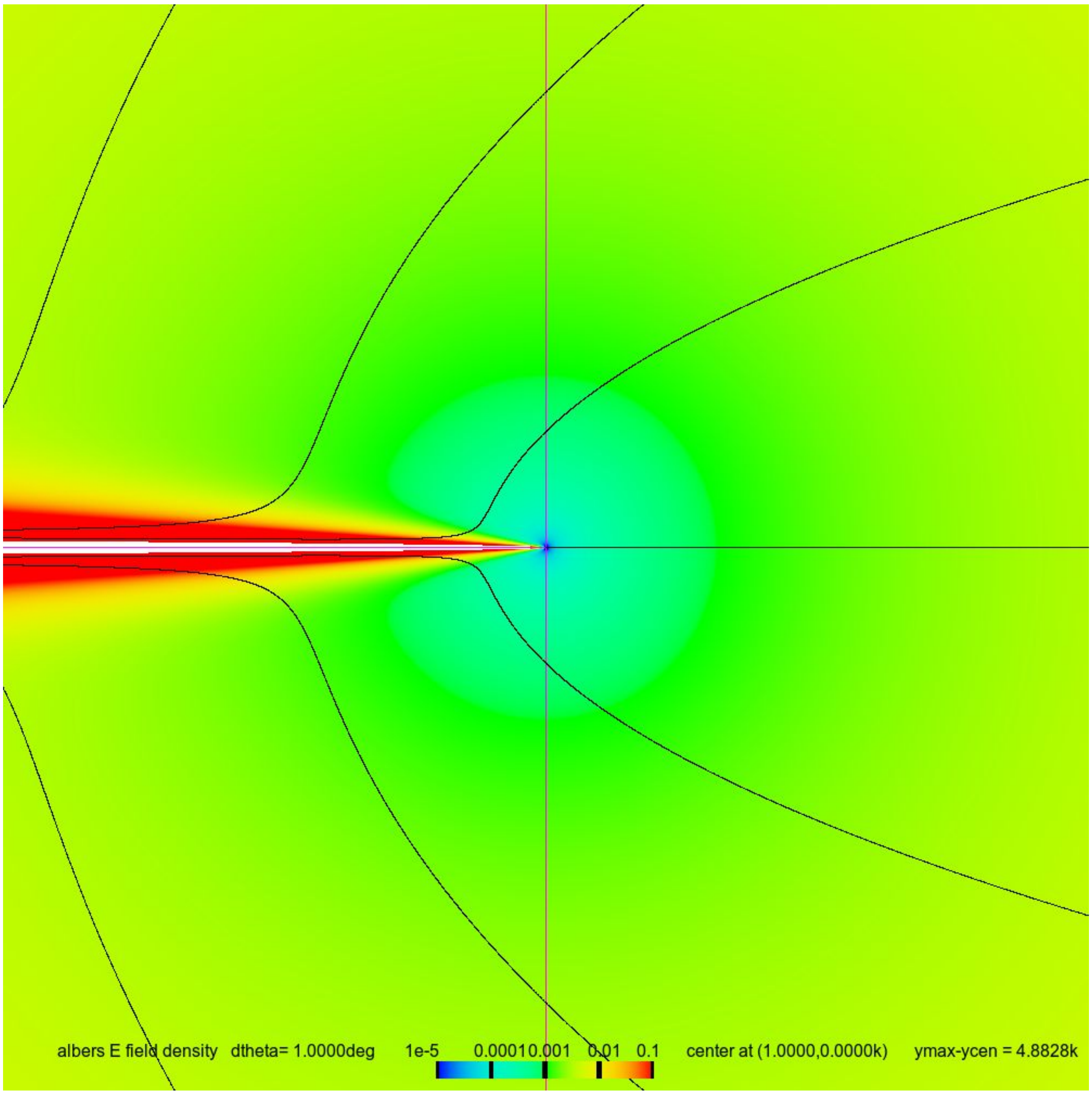

Fig 2: Same as Fig. 1, except zoomed in at the Kerr ring edge at x=+1, y=0. Y varies between +/- 4.8k.

Figure (2) shows a greatly magnified plot at the edge of the Kerr ring, source of 'charge'. *k* is (1/137) x (1/22.5). Relationships between various distance parameters are summarized as follows:

$$k = r_e = \frac{a}{(22.5/\alpha)} \sim a/3000\,, \tag{17}$$

where *a* is the Kerr radius and α is the fine structure constant.

4. Conclusion

Before these plots were done correctly in November, 2016, I was pursuing all analytic means available to see and understand what was being expressed in mathematics. One can plot one line at a time, not enough to see the conformations of the field. Two things are clear: the very near-field sweeps at an angle as it nears the source. Scale here is of the order of the energy radius, out to maybe ten times this, 10*k*. This field sweeps just outside the charge/current source, and is in high contradistinction to the DKS fields [1], whose θ-component dies away close in. Those fields meet the source perpendicularly. Second, one is struck by the overall smoothness and circularity of field intensity, also extremely different from DKS fields, with their null point at *r* =1 on the spin axis.

References:

1. Kerr (1963) ...
2.. Debney, Kerr, Schild, Solution of the Einstein and Einstein-Maxwell equations, *Journal of Mathematical Physics*, Vol. 10, number 10, Oct. 1969
3. . http://stevealbers.net/physics/efield/efield.html
4. Adler, Bazin, Schiffer, Introduction to General Relativity, McGraw-Hill (1975) p.411
5. Personal e-mail communication.

APPENDIX A

Integration of radial DKS field energy starts by constructing:

$$E^r E_r = (r^2 + 1)/(r^2 + c^2)\,[(-r^2 + c^2)^2/(r^2 + c^2)^2\,]\,.$$

The integral reads: $$\xi = \int_r^\infty dr \int_0^1 dc\, (r^2 + 1)(r^4 + c^4 - 2r^2c^2)/(r^2 + c^2)^4\,.$$

The custom here has been covering one hemisphere of integration with cos(θ) going from 0 to 1. We can make a useful rearrangement of the second parenthesized term, if we add and subtract $2r^2c^2$: (...) = $(r^4 + c^4 + 2r^2c^2 - 4r^2c^2)$, since now the first three terms equal the square of $(r^2 + c^2)$. We can do the parts integrations in *c* if we continue: $\int_0^1 dc\ [1/(r^2+c^2)^2 - 4r^2c^2/(r^2+c^2)^4]$.

The same trick, this time adding and subtracting $4c^4$, gives a nicely workable form:

$$\int_0^1 dc[1/(r^2+c^2)^2 - 4c^2/(r^2+c^2)^3 + 4c^4/(r^2+c^2)^4 \ .$$

The two higher-order denominator terms will blend into the first one. Each parts integration takes two orders of *c* to reduce the denominator order by one, as follows:

$4\int_0^1 dc\ c^4/(r^2+c^2)^4$ uses $u = c^3,\ dv = cdc/(r^2+c^2)^4$,

whence $du = 3c^2dc,\ v = -1/(3 \bullet 2)(r^2+c^2)^{-3}$.

The first "*uv*" term is: $uv|^1_0 = (1/2)/(r^2+1)^3$ when evaluated over the limits in *c*. Such a term will contribute not much when integrated by *r* . The inner radius limit is much less than 1, since in this theory, it is taken as the fine structure constant. The significant integral term is now:

$$(4/2)\int_0^1 dc[c^2/(r^2+c^2)^3] \ ,$$

which is the same form as the second term we start with. Adding the two, we can write:

$$\int_0^1 dc\ [1/(r^2+c^2)^2 - 2c^2/(r^2+c^2)^3] \ .$$

The same process will reduce the second term to be like the first ! We get an important factor from this parts integration, so that our total in terms of the first term, is:

$$½ \int_o^1 dc\ 1/(r^2+c^2)^2 \ .$$

Now we fetch the CRC integral tables, #48, and analyze to get:

$$(1/4r^2)\int_0^1 dc\ /(r^2+c^2) = (1/4r^2)\ [1/(r^2+1) + (1/r)\ tan^{-1} 1/r] .$$

In the inner limit for integration over *r,* the remaining step, the arctan is close to $\pi/2$, and once again we may ignore small *r* compared to 1, and write:

$$\xi = 1/4\int dr\ [1/r^2 + 1/r^3\ \pi/2] = 1/4\ [1/r + \pi/(4r^2)].$$

This is the result of integrating by *r* and by θ, as we understand an implied 2π with the final angular integration. The point is comparing this with the simple integration of a one over r-squared field, which yields here a value of 1/r, leaving out the 2π. Given a hemispheric treatment we expect half the usual value, but this result does not yield that!